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INTEGRO-DIFFERENTIAL METHOD OF SOLVING THE INVERSE COEFFICIENT HEAT CONDUCTION PROBLEM

V. L. Baranov, A. A. Zasyad'ko, and G. A. Frolov

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On the basis of differential transformations, a stable integro-differential method of solving the inverse heat conduction problem is suggested. The method has been tested on the example of determining the thermal diffusivity on quasi-stationary fusion and heating of a quartz glazed ceramics specimen.

Keywords: *inverse heat conduction problem, differential transformations, thermophysical characteristics, thermoprotective material, analytical model.*

The basic methods of determining the thermophysical characteristics, as well as of many other engineering and technological parameters of various processes, are methods based on the solution of inverse problems.

The main problem arising in solving inverse problems consists in computational instability caused by the ill-posedness of such problems [1]. Various methods are suggested to obtain stable solutions [2]. For example, in a quasi-stationary regime of heating and entrainment of the mass of a heat protective material, when to recover the thermal conductivity coefficient $\lambda(T)$ it is sufficient to measure the temperature at one point of a specimen, the needed stability of the solution of the inverse problem is ensured by the choice of the temperature interval (step) in which $\lambda(T)$ is determined. Here, the algorithm for selecting the temperature interval is regularizing in essence. To obtain stable solutions of inverse problems, A. N. Tikhonov's regularization method [1] is most often used, which is reduced to the solution of a variational problem that requires a large volume of calculations.

In [3], a solution of an inverse coefficient heat conduction problem (ICHCP) is considered on the assumption that it has only one solution. However, such an assumption does not simplify the computational complexity of its solution, since the ICHCP is reduced to the solution of an optimal control problem related to the class of variational problems whose solution needs a large amount of machine calculations.

In the present work, using as an example the solution of an ICHCP for the temperature field in a material with constant thermophysical properties, an approach allowing one to considerably reduce the amount of computations for determining thermophysical characteristics is suggested.

Let us consider a linear heat conduction equation of the form

$$\frac{\partial T}{\partial t} = a\,\frac{\partial^2 T}{\partial y^2} \tag{1}$$

with boundary conditions allowing for mass entrainment from the material surface at a constant velocity:

$$T(y, t)\big|_{y=S(t)} = T_{\rm w}\,, \tag{2}$$

$$-\left(\lambda\,\frac{\partial T}{\partial y}\right)\Bigg|_{y=S(t)} = \rho c \overline{V}_{\infty}\,(T_{\rm w} - T_0)\,. \tag{3}$$

I. V. Frantsevich Institute for Problems of Material Science, National Academy of Sciences of Ukraine, 3 Krzhizhanovskii Str., Kiev, 03142, Ukraine; email: g_frolov@nbi.com.ua. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 83, No. 1, pp. 54–63, January–February, 2010. Original article submitted August 13, 2008; revision submitted June 7, 2009.

In [4], an expression for calculating the temperature field was obtained in the form

$$T(y,t) = T_0 + (T_w - T_0)\exp\left[-\frac{\overline{V}_\infty}{a}(y - \overline{V}_\infty t)\right]. \tag{4}$$

Equation (4) cannot be used for determining the coefficient a because of its computational instability. With the aid of the Laplace transformation and Fourier convolution, we represent the solution of Eq. (1) in an integral form:

$$T(y,t) = \sum_{j=1}^{m} \int_0^t \int_{S(t)}^{y_s} G_j(t-\tau, y-\xi)\, g_j(\tau,\xi)\, d\xi d\tau . \tag{5}$$

In [5], the Poison kernel for the Dirichlet problem was calculated:

$$G_1(y,t) = (2\sqrt{a\pi})^{-r}\, t^{\frac{r+2}{2}} \exp\left(-\frac{|y|^2}{4at}\right).$$

On substitution of the dimensionality of the problem ($r = 1$), a kernel of the following form can be obtained:

$$G_1(y,t) = \frac{1}{2\sqrt{a\pi t^3}} \exp\left(-\frac{|y|^2}{4at}\right).$$

For the Neumann problem

$$G_2(y,t) = -\,2a\,(2\sqrt{a\pi t})^{-r} \exp\left(-\frac{y^2}{4at}\right),$$

i.e., at $r = 1$, we obtain

$$G_2(y,t) = -\sqrt{\frac{a}{\pi t}} \exp\left(-\frac{y^2}{4at}\right).$$

As a result, we arrive at the expression

$$T(y,t) = \int_{S(t)}^{y_s}\int_0^t \frac{T_w \exp\left(\frac{-(y-\xi)^2}{4a(t-\tau)}\right)}{2\sqrt{a\pi(t-\tau)^3}}\, d\tau d\xi - \int_{S(t)}^{y_s}\int_0^t \frac{(T_w - T_0)\, c\rho \overline{V}_\infty \sqrt{a}\exp\left(\frac{-(y-\xi)^2}{4a(t-\tau)}\right)}{\sqrt{\pi(t-\tau)}}\, d\tau d\xi . \tag{6}$$

According to (4), the left-hand side of Eq. (6) can be presented in the form

$$T(y,t) = T_0 + (T_w - T_0)\exp\left[\frac{-\overline{V}_\infty y}{a} + \frac{\overline{V}_\infty^2 t}{a}\right]. \tag{7}$$

Consequently, the problem (1)–(3) has been reduced to (6), where we have two parts of the equation: on the left is the known function, and on the right the difference of double integrals that takes into account boundary conditions (2) and (3) in Dirichlet and Neumann forms. In Eq. (6) the thermal diffusivity is unknown; it is to be found by solving the inverse heat conduction problem.

We present the method of finding thermophysical characteristics, in particular the thermal diffusivity, on the basis of solving Eq. (6). The method is based on application of differential transformations (introduced in [6]) to the right-hand side of Eq. (6). This allows one to substantially simplify the initial expression due to the obtained approximations at the expansion points. For this purpose, we apply differential transformations first to the first integral and then to the second. The method of differential transformations and its basic properties are considered in considerable detail in [4].

Construction of the Differential-Taylor Model (DT model) of Heat Conduction Problem. *A. Construction of approximating functions for the first integral of Eq.(6).*:

$$I_1 = \int_{\xi_1}^{\xi_2} \int_{\tau_1}^{\tau_2} \frac{T_w \exp\left(\frac{-(y-\xi)^2}{4a(t-\tau)}\right)}{2\sqrt{a\pi(|t-\tau|)^3}} d\tau d\xi = D_1 \int_{\xi_1}^{\xi_2} \int_{\tau_1}^{\tau_2} \frac{\exp\left(\frac{-(y-\xi)^2}{4a(t-\tau)}\right)}{\sqrt{(|t-\tau|)^3}} d\tau d\xi , \quad D_1 = \frac{T_w}{2\sqrt{a\pi}} . \tag{8}$$

The integral I_1 will be considered successively at ξ = const for the internal integral and at τ = const for the external one. The internal integral can be presented as follows:

$$I_{1\text{int}} = \int_{\tau_1}^{\tau_2} \frac{\exp[\nu(\tau)]}{\sqrt{(|t-\tau|)^3}} d\tau = \int_{\tau_1}^{\tau_2} \frac{u(\tau)}{t_1(\tau)} d\tau = \int_{\tau_1}^{\tau_2} n_1(\tau)\, u(\tau)\, d\tau , \tag{9}$$

where $u(\tau) = \exp[(\nu(\tau)]$; $\nu(\tau) = \frac{-(y-\xi)^2}{4a(t-\tau)}$; $n_1(\tau) = \frac{1}{t_1(\tau)} = \frac{1}{\sqrt{(|t-\tau|)^3}}$.

1. We introduce the subradical expression:

$$e_1(\tau) = (t-\tau)^3 = t^3 - 3t^2\tau + 3t\tau^2 - \tau^3 .$$

2. We transfer the expression obtained into the region of differential Taylor transforms [6]:

$$E_1(k) = t^3 ъ(k) - 3t^2 H ъ(k-1) + 3tH^2 ъ(k-2) - H^3 ъ(k-3) , \tag{10}$$

where k = 0, 1, 2, 3, ... and $ъ(k) = \begin{cases} 1 , & k = 0 ; \\ 0 , & k \geq 1 . \end{cases}$ Some of the discrete steps from (10) are defined as $E_1(0) = t^3$; $E_1(1) = -3t^2H$; $E_1(2) = 3tH^2$; $E_1(3) = -H^3$; $E_1(k \geq 4) = 0$.

3. The denominator is $t_1(\tau) = \sqrt{e_1(\tau)} = \sqrt{(|t-\tau|)^3}$. According to [6], the recurrent formula for obtaining the discrete steps of the root has the form

$$T_1(k) = \sqrt[2]{E_1(k)} = \frac{E_1(k) - \sum_{l=1}^{k} \sqrt[2]{E_1(l)}\, \sqrt[2]{E_1(k-l)}}{\sqrt{E_1(0)}} . \tag{11}$$

Then the first three discrete steps will be written as

$$T_1(0) = \sqrt[2]{E_1(0)} = \sqrt{E_1(0)} = \sqrt{t^3} , \quad T_1(1) = \sqrt[2]{E_1(1)} = \frac{E_1(1)}{2\sqrt{E_1(0)}} = \frac{-3H\sqrt{t}}{2} ,$$

$$T_1(2) = \sqrt[2]{E_1(2)} = \frac{E_1(2) - (\sqrt[2]{E_1(1)})^2}{2\sqrt{E_1(0)}} = \frac{3H^2}{8\sqrt{t}} .$$

Consequently, with the aid of reverse transformations of the differential spectrum [6] the denominator $t_1(\tau) = \sqrt{(|t-\tau|)^3}$ can be presented in the form $\sqrt{t^3} - \frac{3}{2}\tau\sqrt{t} + \frac{3\tau^2}{8\sqrt{t}}$.

4. We transfer the fraction $n_1(\tau) = \dfrac{1}{t_1(\tau)}$ into the region of DT transforms:

$$N_1(k) = \left| \frac{ъ(k)}{T_1(k)} \right| = \frac{ъ(k) - \sum_{l=1}^{k} T_1(l)\, N_1(k-l)}{T_1(0)} . \tag{12}$$

Then the first three discrete steps $N_1(k)$ will be written as

$$N_1(0) = \frac{1}{T_1(0)} = \frac{1}{\sqrt{t^3}}, \quad N_1(1) = -\frac{T_1(1)\, N_1(0)}{T_1(0)} = \frac{3H\sqrt{t}}{2t^3},$$

$$N_1(2) = -\frac{T_1(1)\, N_1(1) + T_1(2)\, N_1(0)}{T_1(0)} = \frac{15H^2}{8t^3\sqrt{t}} .$$

Consequently, on the basis of inverse DT transformations the fraction $n_1(\tau) = \dfrac{1}{\sqrt{t-\tau^3}}$ can be represented by the function $\frac{1}{\sqrt{t^3}} + \frac{3\tau\sqrt{t}}{2t^3} + \frac{15\tau^2}{8t^3\sqrt{t}}$. It remains to determine the numerator of the integrand $u(\tau)$.

5. We introduce the notation for the exponent:

$$\nu(\tau) = \frac{-(y-\xi)^2}{4a(t-\tau)} = \frac{B}{t-\tau}, \quad B = t\nu(\tau) - \nu(\tau)\,\tau = \frac{-(y-\xi)^2}{4a},$$

and transfer it into the region of DT transforms:

$$tV(k) - HV(k-1) = Bъ(k),$$

whence we obtain the discrete steps of the differential spectrum:

$$V(0) = B/t, \quad V(1) = HV(0)/t, \quad V(2) = HV(1)/t .$$

On the basis of the inverse DT transformations, $\nu(\tau) = \dfrac{B}{t-\tau}$ can be represented by the function

$$\nu(\tau) = \frac{B}{t} + \frac{B\tau}{t^2} + \frac{B\tau^2}{t^3} . \tag{13}$$

6. Let us find the function exp $[\nu(\tau)]$. To determine the discrete steps of the T function $\underline{\exp}\,[V(k)]$ on the assumption that the discrete steps of the T function $V(k)$ are known, we will avail ourselves of the recurrent formula [6] ($\underline{\exp}$ denotes the DT transform of the exponential function):

$$\underline{\exp}\, V(k+1) = \sum_{l=0}^{l=k} \frac{l+1}{k+1}\, V(l+1)\, \underline{\exp}\,(k-l),$$

or with account for designations (9)

$$U\,(k+1)=\sum_{l=0}^{l=k}\frac{l+1}{k+1}\,V\,(l+1)\,U\,(k-l)\,,\quad \underline{\exp}\,[V\,(0)]=\underline{\exp}\,[v\,(0)]\,. \tag{14}$$

from (14) we find individual discrete steps of the kernel:

$$U\,(0)=\exp\left(\frac{B}{t}\right),\quad U\,(1)=V\,(1)\,U\,(0)\,,\quad U\,(2)=0.5V\,(1)\,U\,(1)+V\,(2)\,U\,(0)\,.$$

Then, on the basis of the inverse DT transforms, the function of the exponent $\exp\,[\nu(\tau)]=\exp\left(\frac{B}{t-\tau}\right)$ is represented as

$$\exp\left(\frac{B}{t}\right)\left(1+\frac{B\tau}{t^2}+0.5\,\frac{B^2\tau^2}{t^4}+\frac{B\tau^2}{t^3}\right)=\exp\left(\frac{B}{t}\right)\left[1+\frac{B\tau}{t^2}+\frac{B\tau^2}{t^3}\left(\frac{B}{2t}+1\right)\right]. \tag{15}$$

7. We designate the integrand as

$$z_1\,(\tau)=\frac{\exp\left(\dfrac{B}{t-\tau}\right)}{\sqrt{(|t-\tau|)^3}}=n_1\,(\tau)\,u\,(\tau)\,.$$

Then the products in the region of transforms (T product) has the form

$$Z_1\,(k)=U\,(k)*N_1\,(k)=\sum_{l=0}^{l=k}U\,(k-l)\,N_1\,(l)\,, \tag{16}$$

where * is the symbol of the operation of multiplication in the region of transforms. Individual discrete steps of the integrand are determined from Eq. (16):

$$Z_1\,(0)=U\,(0)\,N_1\,(0)=\frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t^3}}\,.$$

$$Z_1\,(1)=U\,(1)\,N_1\,(0)+U\,(0)\,N_1\,(1)=\frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t^3}}\left(\frac{BH}{t^2}+\frac{3H}{2t}\right),$$

$$Z_1\,(2)=U\,(2)\,N_1\,(0)+U\,(1)\,N_1\,(1)+U\,(0)\,N_1\,(2)=\frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t^3}}\,\frac{H^2}{8t^4}\,(15t^2+20Bt+4B^2)\,.$$

On the basis of the inverse DT transformations, we find the approximation of the integrand $z_1(\tau)$ in the form

$$\frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t^3}}\left[1+\frac{\tau}{t}\left(\frac{B}{t}+\frac{3}{2}\right)+\frac{\tau^2}{8t^4}\,(15t^2+20Bt+4B^2)\right].$$

8. The definite integral over the integration variable s within the limits $\alpha_c - \beta_c$ of the function $z_1(s)$ over its discrete steps in the region of transforms $Z_1(0)$, $Z_1(1)$, $Z_1(2)$, ..., $Z_1(\infty)$ is determined according to [6] from the expression

$$\int\limits_{\alpha_c}^{\beta_c} z_1(s)\, ds = \sum_{k=0}^{k=\infty} \left(\frac{\beta_c^{k+1} - \alpha_c^{k+1}}{k+1} \right) \frac{Z_1(k)}{H^k} = \frac{\beta_c - \alpha_c}{1} Z_1(0) + \frac{\beta_c^2 - \alpha_c^2}{2H} Z_1(1) + ..., \tag{17}$$

where $\alpha = \alpha_c - s_0$, $\beta = \beta_c - s_0$, s_0 is the center of expansion into a Taylor series. In a particular case $s_0 = 0$.

Consequently, the internal integral (9) that takes into account three discrete steps is expressed by the function

$$I_{1\text{int}} = \frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t^3}} \left[\beta_1 - \alpha_1 + \frac{\beta_1^2 - \alpha_1^2}{2} \left(\frac{B}{t^2} + \frac{3}{2t} \right) + \frac{\beta_1^3 - \alpha_1^3}{24t^4} (15t^2 + 20Bt + 4B^2) \right]. \tag{18}$$

In the case of the account for the zero and first discrete steps the internal integral (9) has the form

$$I_{1\text{int}} = \frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t^3}} \left[\beta_1 - \alpha_1 + \frac{\beta_1^2 - \alpha_1^2}{2} \left(\frac{B}{t^2} + \frac{3}{2t} \right) \right], \quad A_1 = \frac{\beta_1 - \alpha_1}{\sqrt{t^3}} + \frac{3(\beta_1^2 - \alpha_1^2)}{4t\sqrt{t^3}}, \quad A_2 = \frac{\beta_1^2 - \alpha_1^2}{8at^2\sqrt{t^3}}. \tag{19}$$

Then

$$I_{1\text{int}} = \exp\left(\frac{-(y-\xi)^2}{4at} \right) (A_1 - (y-\xi)^2 A_2) = \exp\,[A_3 (y-\xi)^2]\,(A_1 - A_2 y^2 + 2A_2 y\xi - A_2\xi^2) = u'\,[w'(\xi)]\, s(\xi)\,, \tag{20}$$

where $A_3 = -\frac{1}{4at}$; $u'(\xi) = \exp\,[w'(\xi)]$; $w'(\xi) = A_3(y-\xi)^2 = A_3(y^2 - 2y\xi + \xi^2)$; $s(\xi) = (A_1 - A_2y^2 + 2A_2y\xi - A_2\xi^2)$.

9. The DT transform for the argument of the exponent $w'(\xi)$ will be found on carrying out differential transformations over the variable ξ:

$$W'(k) = A_3\,[y^2 \text{ъ}(k) - 2yH \text{ъ}(k-1) + H^2 \text{ъ}(k-2)]\,. \tag{21}$$

Then the discrete steps are defined as follows: $W(0) = A_3y^2$, $W'(1) = -2A_3yH$, $W'(2) = A_3H^2$, $W'(k \geq 3) = 0$.

10. The transforms of the exponent are found from (14), and individual discrete steps of the exponent have the form

$$U'(0) = \exp\,(A_3y^2)\,, \quad U'(1) = -\,2A_3yH \exp\,(A_3y^2)\,,$$

$$U'(2) = 2A_3^2y^2H^2 \exp\,(A_3y^2) + A_3H^2 \exp\,(A_3y^2) = A_3H^2 \exp\,(A_3y^2)\,(2A_3y^2 + 1)\,.$$

Using the inverse transformations for the argument ξ, we obtain the inverse transform of the exponent:

$$\exp\,[A_3\,(y-\xi)^2] = \exp\,(A_3y^2)\,(1 - 2A_3y\xi + A_3\xi^2\,(2A_3y^2 + 1))\,.$$

11. We transfer the multiplier $s(\xi) = (A_1^2 - A_2y^2 + 2A_2y\xi - A_2\xi^2)$ in (20) into the DT transform by the argument ξ similarly to (10). As a result, we obtain the discrete steps

$$S(0) = A_1^2 - A_2y^2\,; \quad S(1) = 2A_2yH\,; \quad S(2) = -\,A_2H^2\,.$$

12. We find the product $z'(\xi) = u'(\xi)s(\xi)$ in the region of DT transforms from (16). Individual discrete steps of the integrand $Z'(k)$ ($k = 0, 1, 2$) over ξ have the form

$$Z'(0) = \exp(A_3 y^2)(A_1 - A_2 y^2),$$

$$Z'(1) = \exp(A_3 y^2)\, 2A_2 yH - 2A_3 yH \exp(A_3 y^2)(A_1 - A_2 y^2) = 2\exp(A_3 y^2)\, Hy\,[A_2 - A_3(A_1 - A_2 y^2)],$$

$$Z'(2) = A_3 H^2 \exp(A_3 y^2)(2A_3 y^2 + 1)(A_1 - A_2 y^2) - 4A_2 A_3 y^2 H^2 \exp(A_3 y^2) - A_2 H^2 \exp(A_3 y^2)$$

$$= H^2 \exp(A_3 y^2)\,[A_3(2A_3 y^2 + 1)(A_1 - A_2 y^2) - 4A_2 A_3 y^2 - A_2].$$

13. With the aid of (17) the external integral (8) on two discrete steps over τ and three discrete steps over ξ will be expressed by the function

$$I_{1\text{ext}} = D_1 \Big\{ (\beta_2 - \alpha_2)\exp(A_3 y^2)(A_1 - A_2 y^2) + (\beta_2^2 - \alpha_2^2)\exp(A_3 y^2)\, y\,(A_2 - A_3(A_1 - A_2 y^2))$$

$$+ \frac{\beta_2^3 - \alpha_2^3}{3}\exp(A_3 y^2)\,[A_3(2A_3 y^2 + 1)(A_1 - A_2 y^2) - 4A_2 A_3 y^2 - A_2] \Big\}. \tag{22}$$

If we take into account only the zero and first discrete steps over ξ, the external integral has the form

$$I_{1\text{ext}} = D_1 \exp(A_3 y^2)\Big\{(\beta_2 - \alpha_2)(A_1 - A_2 y^2) + (\beta_2^2 - \alpha_2^2)\, y\,(A_2 - A_3(A_1 - A_2 y^2))\Big\}. \tag{23}$$

Allowance for only zero discrete steps yields the following expression for $I_{1\text{ext}}$:

$$I_{1\text{ext}} = D_1 \frac{\exp\left(\frac{-y^2}{4at}\right)}{\sqrt{t^3}}(\beta_2 - \alpha_2)(\beta_1 - \alpha_1). \tag{24}$$

When only one (zero) discrete step over ξ and three discrete steps over τ are taken into account, we obtain $I_{1\text{ext}}$ in the form

$$I_{1\text{ext}} = D_1 \frac{\exp\left(\frac{-y^2}{4at}\right)}{\sqrt{t^3}}(\beta_2 - \alpha_2)\left[\beta_1 - \alpha_1 + \frac{\beta_1^2 - \alpha_1^2}{2}\left(\frac{B}{t^2} + \frac{3}{2t}\right) + \frac{\beta_1^3 - \alpha_1^3}{24t^4}(15t^2 + 20Bt + 4B^2)\right]. \tag{25}$$

B. Construction of approximation functions for the second integral of expression (6):

$$I_2 = \int_{\xi_1}^{\xi_2}\int_{\tau_1}^{\tau_2} \frac{\rho \bar{V}_\infty c\,(T_w - T_0)\sqrt{a}}{\sqrt{\pi}\,|t-\tau|}\exp\left[\frac{-(y-\xi)^2}{4a(t-\tau)}\right] d\tau d\xi = D_2 \int_{\xi_1}^{\xi_2}\int_{\tau_1}^{\tau_2} \frac{\exp\left[\frac{-(y-\xi)^2}{4a(t-\tau)}\right]}{\sqrt{|t-\tau|}}\, d\tau d\xi, \tag{26}$$

where

$$D_2 = \frac{\rho \bar{V}_\infty c\,(T_w - T_0)\sqrt{a}}{\sqrt{\pi}}.$$

Let $n_2(\tau) = \dfrac{1}{\sqrt{|t-\tau|}} = \dfrac{1}{t_2(\tau)}$, $e_2(\tau) = t - \tau$. The DT transform of the subradical expression $e_2(\tau)$ is

$$E_2(k) = t\text{ъ}(k) - H\text{ъ}(k-1)\,,\quad E_2(0) = t\,,\quad E_2(1) = -H\,,\quad E_2(k \geq 2) = 0\,.$$

The DT transform of the root $t_2(\tau) = \sqrt{|t-\tau|}$ in (26) is defined by (11):

$$T_2(0) = \sqrt[2]{E_2(0)} = \sqrt{E_2(0)} = \sqrt{t}\,,\quad T_2(1) = \sqrt[2]{E_2(1)} = \frac{E_2(1)}{2\sqrt{E_2(0)}} = \frac{-H}{2\sqrt{t}}\,,$$

$$T_2(2) = \sqrt[2]{E_2(2)} = \frac{E_2(2) - (\sqrt[2]{E_2(1)}\,)^2}{2\sqrt{E_2(0)}} = \frac{-H^2}{8t\sqrt{t}}\,.$$

The inverse transformation of the transform of the denominator $\sqrt{|t-\tau|}$ yields its inverse transform in the form $\sqrt{t} - \dfrac{\tau}{2\sqrt{t}} - \dfrac{\tau^2}{8t\sqrt{t}}$.

The DT transform of the fraction $n_2(\tau)$ in (26) is defined by (12):

$$N_2(0) = \frac{1}{T_2(0)} = \frac{1}{\sqrt{t}}\,,\quad N_2(1) = -\frac{T_2(1)\,N_2(0)}{T_2(0)} = \frac{H}{2t\sqrt{t}}\,,$$

$$N_2(2) = -\frac{T_2(1)\,N_2(1) + T_2(2)\,N_2(0)}{T_2(0)} = \frac{3H^2}{8t^2\sqrt{t}}\,.$$

On the basis of the inverse DT transforms we obtain the approximation of the denominator in (26) in the form

$$|t-\tau|^{-\frac{1}{2}} \approx \frac{1}{\sqrt{t}} + \frac{\tau}{2t\sqrt{t}} + \frac{3\tau^2}{8t^2\sqrt{t}}\,.$$

The numerator of the integrand in (26) $u(\tau) = \exp\,[\nu(\tau)]$ is calculated from (13), (14) and will have the form of (15). Let us introduce the designations of the integrand in (26) $z_2(\tau) = n_2(\tau)u(\tau)$. Then the product in the region of transforms (T product) for the integrand is calculated from (16). For the zero and first discrete steps we obtain the expressions

$$Z_2(0) = \frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t}}\,,\quad Z_2(1) = \frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t}}\left(\frac{H}{2t} + \frac{BH}{t^2}\right).$$

The definite integral within $\alpha - \beta$ of the function $z_2(\tau)$ over its discrete steps in the region of the transforms $Z_2(0)$, $Z_2(1)$ can be found from (17):

$$I_{2\text{int}} = \frac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t}}\left[\beta_1 - \alpha_1 + \frac{\beta_1^2 - \alpha_1^2}{2}\left(\frac{1}{2t} + \frac{B}{t^2}\right)\right].$$

Let $A_{21} = \dfrac{\beta_1 - \alpha_1}{\sqrt{t}} + \dfrac{\beta_1^2 - \alpha_1^2}{2}\dfrac{1}{2t\sqrt{t}}$, $A_{22} = \dfrac{\beta_1^2 - \alpha_1^2}{8at^2\sqrt{t}}$. With account for the designations introduced, we obtain

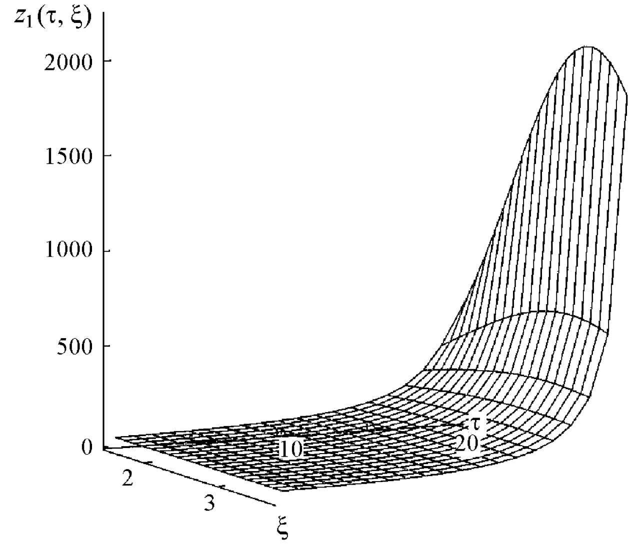

Fig. 1. Integrand $z_1(\tau, \xi) = \dfrac{\exp\left[\dfrac{(y-\xi)^2}{4a(t-\tau)}\right]}{\sqrt{(|t-\tau|)^3}}$.

$$I_{2\text{int}} = \exp\left[\frac{-(y-\xi)^2}{4at}\right](A_{21} - (y-\xi)^2 A_{22}) = \exp\,[A_3\,(y-\xi)^2]\,(A_{21} - A_{22}y^2 + 2A_{22}y\xi - A_{22}\xi^2)\,,\quad A_3 = -\frac{1}{4at}\,.$$

This expression coincides with Eq. (20). Since the equation to calculate the external integral $I_{2\text{ext}}$ coincides with the equation for the $I_{1\text{ext}}$ integral, the final representation has the form of (22):

$$I_{2\text{ext}} = D_2\left\{(\beta_2 - \alpha_2)\exp\,(A_3y^2)\,(A_{21} - A_{22}y^2) + (\beta_2^2 - \alpha_2^2)\exp\,(A_3y^2)\,y\,(A_{22} - A_{23}\,(A_{21} - A_{22}y^2))\right.$$
$$\left. + \frac{\beta_2^3 - \alpha_2^3}{3}\exp\,(A_3y^2)\,[A_3\,(2A_3y^2 + 1)\,(A_{21} - A_{22}y^2) - 4A_{22}A_3y^2 - A_{22}]\right\}. \quad (27)$$

Thus, approximate expressions (22) and (27) have been obtained to calculate the integrals on the right-hand side of Eq. (6). With allowance for (7) Eq. (6) has the form

$$T_0 + (T_\text{w} - T_0)\exp\left[\frac{-\bar{V}_\infty y}{a} + \frac{\bar{V}_\infty^2 t}{s}\right] = I_1 - I_2\,, \quad (28)$$

where I_1 is calculated approximately from (22), and I_2 from (27). To reduce the amount of calculations of the I_1 integral less exact equations (23)–(25) can be used. It is worthwhile to use them to obtain the primary estimate of the thermal diffusivity by solving Eq. (28).

The integrand $z_1(\tau, \xi)$ is shown in Fig. 1. It is seen that the surface of $z_1(\tau, \xi)$ relative to ξ has insignificant differences and increases rapidly over the time coordinate with increase in the time of heating. As to the form of $z_1(\tau, \xi)$, it should be noted that the number of the points of division over the time coordinate for the DT model should be much larger than over the spatial coordinate.

Figure 2 presents different approximations $z_1(\tau, \xi)$ by displaced DT transformations at the points t_ν, τ_ν over the time coordinates with a step $h_{1\nu}$ and $h_{2\nu}$, respectively ($t_\nu = t - h_{1\nu}$, $\tau_\nu = \tau - h_{2\nu}$).

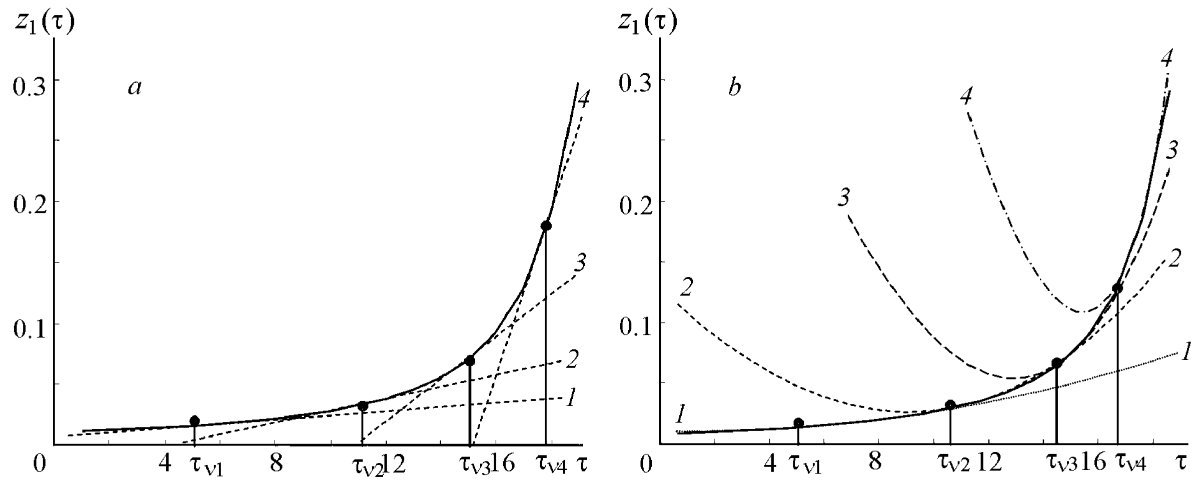

Fig. 2. Representation of the $z_1(\tau) = \dfrac{\exp\left[\dfrac{B}{(t-\tau)}\right]}{\sqrt{(|t-\tau|)^3}}$ (solid line) at the points τ_v by approximations of $z_1(\tau)\Big|_{\substack{\tau=\tau_n \\ t=t_v}} = \dfrac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t^3}}\left[1+\dfrac{\tau}{t}\left(\dfrac{B}{t}+\dfrac{3}{2}\right)\right]$ (a) and $z_1(\tau)\Big|_{\substack{\tau=\tau_n \\ t=t_v}} = \dfrac{\exp\left(\dfrac{B}{t}\right)}{\sqrt{t^3}}\left[1+\dfrac{\tau}{t}\left(\dfrac{B}{t}+\dfrac{3}{2}\right)+\dfrac{\tau^2}{8t^4}(15t^2+20Bt+4B^2)\right]$ (b): 1–4, approximation at points $\tau_{v1}-\tau_{v4}$.

TABLE 1. Influence of δ_{model} and δ_y on the Relative Error of Determining the Thermal Diffusivity δ_a for Models (7) and (28)

Values of parameters n and p	δ_{model}, %	δ_y, %	$a\cdot10^7$, m^2/sec	δ_a, %
Analytical model (7)				
—	0	0	6	0
—	0	12.5	7.68	28
Integral-differential model (28)				
n = 3, p = 2	8.9	0	2.2	63
n = 12, p = 1	3	0	5.75	4.17
n = 12, p = 1	3	12.5	5.12	14.67
n = 21, p = 1	0.05	0	5.87	2.17
n = 21, p = 1	0.05	12.5	5.23	12.83

In the case of uniform division of the time interval into n parts and of the spatial coordinate into p parts for the experimental parameters $\overline{V}_\infty = 0.11\cdot10^{-3}$ m/sec, c = 1759 kJ/(kg·K), ρ = 2000 kg/m^3, t = 20 sec, $y = 4\cdot10^{-3}$ m, T_w = 2390 K, $\xi_1 = \alpha_2 = 2.29462722\cdot10^{-3}$ m, $\xi_1 = \beta_2 = 3.807539625\cdot10^{-3}$ m, $\tau_1 = \alpha_1 = 0$ sec, $\tau_2 = \beta_2 = 19$ sec, Eq. (28) yields the following values of residuals: $\varepsilon = I_1 - I_2 - T(y, t)$: $n = 2$, $p = 2$, $\varepsilon = 115.3036$, $n = 3$, $p = 2$, $\varepsilon = -11.9817$, $n = 3$, $p = 5$, $\varepsilon = -6.2504$, $n = 12$, $p = 2$, $\varepsilon = 0.0770$.

Table 1 presents the solution of the inverse heat conduction problem for thermal diffusivity for analytical model (7) and model (28) with approximation of the temperature field distribution over one discrete step for various errors of experimental data. It was found that models (7) and (28) are most sensitive to the depth of heating of the specimens investigated; therefore, the error δ_y was introduced into the parameter y. An analysis of the numerical experiment shows that the analytical solution is sensitive to changes of y in the initial data and as a result the error δ_a of the determination of a considerably exceeds the error in the initial data $\delta_a >> \delta_y$. This points to the instability and, consequently, to the unsuitability of the analytical model for solving the inverse problem. Conversely, the integral-differential model (28) demonstrates stability, since $\delta_a \approx \delta_y$; δ_a decreases with increase in the number of partition points n, p. Attention should be paid to the fact that the integral-differential model (28) has one additional error of model approximation which, as it turned out, does not influence the stability of the solution obtained.

The processing of data has shown that the experimental temperature field obtained in quartz glass ceramics specimens with $a \approx 0.6 \cdot 10^{-6}$ m^2/sec in the considered range of temperatures agrees well with calculation by Eq. (28). The solution of Eq. (28) for a made it possible to obtain its value with an error not exceeding 3%. This confirms the computational stability of the proposed method of solving inverse problems.

The basic result of the present work is in the reduction of the ICHCP, which requires extensive computations, to the solution of a simpler problem consisting in the solution of Eq. (28) for thermal diffusivity. For comparison, we will give the results of the solution of the test ICHCP problem as an optimal control problem on a BESM-6 computer having a speed of response of a million operations per second. According to [3], the time of solving the problem was equal to about 2.5 min, during which $1.5 \cdot 10^8$ operations were performed. In solving (28) the quantity of operations on a modern computer of similar response is several orders smaller. The reduction of the mathematical complexity in solving the ICHCP has a practical importance for controlling and monitoring thermal processes in real time.

Conclusions. With the aid of the integral-differential method based on differential transformations of the heat-conduction problem in an integral-differential form, the solution of the inverse coefficient heat conduction problem with mass entrainment, which is stable against errors in experimental data, has been obtained.

NOTATION

a, thermal diffusivity, m^2/sec; c, heat capacity, kJ/(kg·K); $E_1(k)$, transform of $e_1(\tau)$; $E_2(k)$, transform of $e_2(\tau)$; $\underline{\exp}\ [V(k)]$, transform of the function exp $[\nu(\tau)]$; $G_1(y, t)$, Poisson kernel for the Dirichlet problem; $G_2(y, t)$, Poisson kernel for the Neumann problem; G_j, Poisson kernel; $g_j(\tau, \xi)$, right-hand side of boundary conditions; H, scale factor of differential Taylor transformations; h, step between two displaced points; I_1, integral form of the representation of the Dirichlet problem; $I_{1\text{int}}$, internal integral for the Dirichlet problem; $I_{1\text{ext}}$, external integral for the Dirichlet problem; I_2, integral form of representation of the Neumann problem; $I_{2\text{int}}$, internal integral for the Neumann problem; $I_{2\text{ext}}$, external integral for the Neumann problem; j, summation index; k, l, integer arguments in the region of transforms equal to 0, 1, 2, 3, etc.; m, number of boundary conditions; $N_1(k)$, transform of expression $n_1(\tau)$; $N_2(k)$, transform of expression $n_2(\tau)$; n, p, integer positive numbers; r, dimensionality of problem; $S(t)$, linear entrainment from the surface, m; $S(k)$, transform of expression $s(\xi)$; s_0, center of expansion into a Taylor series; T, temperature, K; $T_1(k)$, transform of expression $t_1(\tau)$; $T_2(k)$, transform of expression $t_2(\tau)$; $T(y, \tau)$, temperature distribution function, K; T_0, temperature of unheated material, K; T_w, temperature of heated surface, K; t, time of heating, sec; t_ν, points of expansion by mixed DT transformations (displaced points) over the time coordinate; $U(k)$, transform of expression $u(\tau)$; $V(k)$, transform of expression $\nu(\tau)$; $\overline{V}_\infty$, quasistationary value of the velocity of linear entrainment of mass, m/sec; $W'(k)$, transform of expression $w'(\xi)$; y, coordinate reckoned from the initial surface, m; y_s, coordinate of the lower boundary of stationary heated layer from the initial surface, m; $Z_1(k)$, transform of function $z_1(\tau)$; $Z_2(k)$, transform of function $z_2(\tau)$; $Z'(k)$, transform of expression $z'(\xi)$ (k = 0, 1, 2); α_c, β_c, limits of integration of function $z_1(s)$; α_1, β_1, lower and upper limits of integration over the variable τ; α_2, β_2, lower and upper limits of integration over the variable ξ; δ_T, stationary value of the depth of heated layer, m; ε, residual of solution; λ, thermal conductivity, W/(m·K); ξ, variable of integration over the coordinate, m; ξ_1, ξ_2, lower and upper limits of integration over ξ; ρ, density, kg/m^3; τ, variable of integration over time, sec; τ_1, τ_2, lower and upper limits of integration over τ; τ_ν, displaced point over time coordinate. Subscripts: c, displaced point over the coordinate; T, temperature; ν, displaced point in time; w, conditions on the wall; δ, heated layer; θ^*, dimensionless temperature of isotherm; 0, unheated material, impenetrable surface.

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